MATH 239 - Introduction to Combinatorics

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Last updated: September 5, 2020

Contents

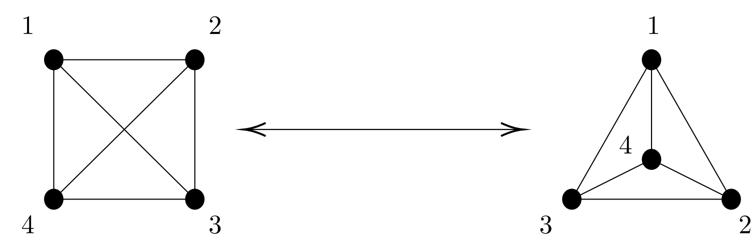
Chapter 7

Planar Graphs

2020-03-23

7.1 Planarity

K_4 is the complete graph on 4 vertices. There are $\binom{4}{2} = 6$ edges.

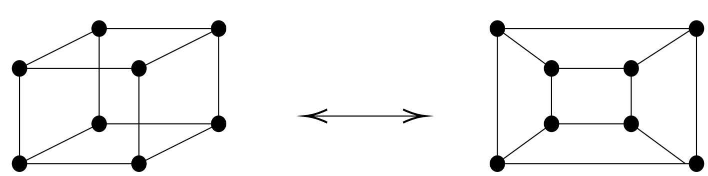


non-planar drawing planar drawing

This picture is not the graph itself, it is a drawing of the graph. There are other drawings of the same graph, as seen on the right.

'The same graph' means that the vertex set and edge set are the same, which implies that they have the same set of adjacent pairs.

Informally, a planar drawing of a graph is a way of drawing the graph in $\mathbb{R}^2$ such that the edges do not cross.



non-planar drawing planar drawing

Clearly, the left is a non-planar drawing as there are 2 crossings within the graph. However, the 3-cube can be drawn in a planar way as seen on the right.

DEFINITION 7.1.1

Informally, a ***planar drawing*** of a graph $G = (V, E)$ is a mapping of the vertices of G to distinct points in $\mathbb{R}^2$, and edges of G to curves between the appropriate pair of vertices in such a way that the edges intersect only at their common ends.

DEFINITION 7.1.2

A graph is ***planar*** if it has a planar drawing.

EXAMPLE 7.1.3: Planar

K_4 and the 3-cube are both planar graphs.

It is **not** correct to say that K_4 is *sometimes* planar depending on how you draw it. It **is** correct to say that K_4 is a planar graph because there exists a planar drawing.

This definition is quite hard to utilize for showing a graph is *not planar*. In fact, there is no obvious way to prove there *exists* a graph that is not planar since there exists 'infinitely' many possible ways to draw a graph.

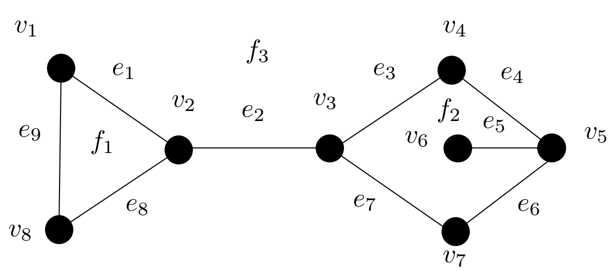
Notice that whether you have a planar drawing, you have *faces*.

Many definitions about graph theory within this course will be informal, as defining them with rigour will be time consuming. Formal definitions can be found in a course like CO 342 (Introduction to Graph Theory), and courses about Topology.

DEFINITION 7.1.4

Given a drawing of a graph, let X be the set of points of $\mathbb{R}^2$ that are part of the drawing. Informally, the ***faces*** of the drawing are the 'connected regions' of $\mathbb{R}^2 \setminus X$.

A planar drawing of a graph is also called an ***embedding***.



Each face of a connected planar drawing has a ***boundary walk***. This is a closed walk of the graph that follows the boundary of the face.

The boundary walk for f_2 is:

$$(v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_5, v_5, e_6, v_7, e_7, v_3)$$

DEFINITION 7.1.5

If an edge e belongs to the boundary walk, of a face f, we say that e is ***incident*** with f.

DEFINITION 7.1.6

If two faces have boundary walks that share an edge, then they are ***adjacent***.

DEFINITION 7.1.7

The ***degree*** of a face is the length of its boundary walk (the number of edges in the walk counting repetitions twice).

EXAMPLE 7.1.8: Degree

In the above graph,
- f_1 has degree 3
- f_2 has degree 6
- f_3 has degree 9

Q: Do the edges in a planar drawing need to be straight lines?

A: No. (THEOREM (Fary ≈ 1950). It doesn't matter).

Q: What about other surfaces than the plane?

A:

- $\mathbb{R}^3$? Too easy.
- Sphere? Interesting.
- Torus?

PROPOSITION 7.1.9

The graphs that can be drawn on a sphere (without crossings) are just the planar graphs. That is, G can be drawn on a sphere (without crossings) if and only if G can be drawn on the plane.

Sketch of *proof*:

($\Longleftarrow$) Obviously true. Planar drawing implies spherical drawing.

($\Longrightarrow$) Take a hole of a balloon, and stretch it (provided the balloon does not burst). Eventually, it will end up as a flat surface.

7.2 Stereographic Projection

Think about a sphere on a flat surface. Shine the light source onto the sphere and a shadow will be cast onto the flat surface.

A 'light source' at $(0, 0, 1)$ casts a 'shadow' of the graph on the sphere as a graph on the plane.

Q: Which graphs can we draw on the torus?

A:

- K_5 is an example of a graph that can be drawn on a torus, but cannot be drawn on a plane.
- K_6?
- K_7?

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Q: Which graphs can be drawn on a mobius strip?

- Non-planar graphs like K_5

THEOREM 7.2.1: Handshaking for Faces

If G is a planar graph and F is the set of faces in some drawing of G, then

$$\sum_{f \in F} \deg(f) = 2|E(G)|$$

Proof of

$$\sum_{f \in F} \deg(f) = \sum_{f \in F} (\text{length of boundary walk of } f)$$

Each edge contributes 1 to the length of exactly two boundary walks of faces (one for each side). Therefore,

$$\sum_{f \in F} (\text{length of boundary walk of } f) = 2|E|$$

In a planar drawing of a tree,

- there is exactly one face
- its degree is $2|E|$

THEOREM 7.2.2: Euler's Formula

If F is the set of faces in a drawing of a connected planar graph $G = (V, E)$, then

$$|V| - |E| + |F| = 2$$

Suppose we have a spanning tree; that is, we have a tree with n vertices, $n-1$ edges, and 1 face. Now,

$$|V|, |E|, |F| \longrightarrow_{+\text{e}} |V|, |E| + 1, |F| + 1$$

Proof of

Suppose for a contradiction that the theorem is false. Let G be a counter-example with as few edges as possible. If G is a tree, then $|E(G)| = |V(G)| - 1$ and any drawing of G has exactly one face, so for any drawing,

$$|V| - |E| + |F| = |V| - (|V| - 1) + 1 = 2$$

So, G is not a counter-example. Otherwise, G is not a tree. Let e be an edge of G that is not a bridge; that is, $G - e$ is connected. Now, a plane drawing of G gives rise to a plane drawing of $G - e$ with exactly one less face. So, the number of faces of (the drawing of) $G - e$ is $|F(G)| - 1$. By the minimality of G, the graph $G - e$ satisfies Euler's Formula. Therefore,

$$|V(G - e)| - |E(G - e)| + (|F(G)| - 1) = 2$$

$$\implies |V(G)| - (|E(G)| - 1) + (|F(G)| - 1) = 2$$

$$\implies |V(G)| - |E(G)| + |F(G)| = 2,$$

contradicting the choice of G.

Let $G = (V, E)$ be a connected planar graph, and F be the set of faces in some drawing of G.

- Handshaking for Faces: $\sum\limits_{f \in F} \deg(f) = 2|E|$.
- Euler's Formula: $|V| - |E| + |F| = 2$.

Q: What are the *connected* planar drawings where:

- every vertex has the same degree ($d \geqslant 3$), and
- every face has the same degree ($k \geqslant 3$)?

A:

- K_4: vertices of degree 3 and faces of degree 3
- cube: vertices of degree 3 and faces of degree 4
- k-cycle: vertices of degree 2 and faces of degree k

7.3 Platonic Solids

DEFINITION 7.3.1

A graph is called ***Platonic*** if it can be drawn in the plane so it is connected, every vertex has degree $d \geqslant 3$, and every face has degree $k \geqslant 3$.

Let G be a Platonic graph. Let $n = |V(G)|$, $m = |E(G)|$, and $\ell = |F|$ (in some plane drawing of G).

By Euler's Formula,

$$n - m + \ell = 2$$

By the Handshake Lemma,

$$2m = \sum_{v \in V(G)} \deg(v) = n \times d$$

By Handshaking for Faces,

$$2m = \sum_{f \in F} \deg(f) = \ell \times k$$

Solving for n and ℓ,

$$n = \frac{2m}{d}$$
$$\ell = \frac{2m}{k}$$

Plugging these into Euler's Formula,

$$2 = \frac{2m}{d} - m + \frac{2m}{k} = m\left(\frac{2}{d} + \frac{2}{k} - 1\right)$$

We have $\frac{2}{d} + \frac{2}{k} - 1 = \frac{2}{m} > 0$. We know $d, k \geqslant 3$. If $d, k \geqslant 4$, then

$$\frac{2}{d} + \frac{2}{k} - 1 \leqslant 0$$

So, one of d, k is at most 3. If (say) $d = 3$, then

$$\frac{2}{d} + \frac{2}{k} - 1 > 0$$

so, $\frac{2}{k} > \frac{1}{3} \implies k < 6$. This only leaves 5 options:

$$(d, k) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$$

The only equation that relates m, k, and d is:

$$\frac{2}{m} = \frac{2}{k} + \frac{2}{d} - 1$$

If $(k,d)=(3,3)$, then

$$\frac{2}{m}=\frac{2}{3}+\frac{2}{3}-1 \implies m=6$$

$$n=\frac{2m}{d}=4,\ \ell=\frac{2m}{k}=4$$

If $(k,d)=(3,3)$, solving gives $n=4,\ell=4,m=6$ which is the ***Tetrahedron*** (K_4); self-dual.

If $(k,d)=(4,3)$, solving gives $n=8,m=12,\ell=6$ which is the ***Cube***; dual: $(k,d)=(3,4)$.

If $(k,d)=(3,4)$, solving gives $n=6,m=12,\ell=8$ which is the ***Octahedron***.

If $(k,d)=(5,3)$, solving gives $n=20,m=30,\ell=12$ which is the ***Dodecahedron***; dual: $(k,d)=(3,5)$.

If $(k,d)=(3,5)$, solving gives $n=12,m=30,\ell=20$ which is the ***Icosahedron***.

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7.4 Non-planar Graphs

In a planar drawing of $G=(V,E)$ with set of faces F, and G connected,

- $|V|-|E|+|F|=2$
- $\sum\limits_{f\in F}\deg(f)=2|E|$

We will combine these to prove that various graphs (such as K_5) are not planar.

PROPOSITION 7.4.1

In any planar drawing of a graph that is not a tree, every face has a boundary walk that contains (the edges of) a cycle.

Therefore, the boundary walk of every face has length $\geqslant 3$ (unless the graph has $\leqslant 1$ edge). It follows that, if G is a connected plane graph with $\geqslant 2$ edges, then in any drawing of G, each face has degree $\geqslant 3$.

PROPOSITION 7.4.2

In a drawing of a connected plane graph $G=(V,E)$ with $|E|\geqslant 2$, and set F of faces, we have

$$|F|\leqslant\frac{2}{3}|E|$$

Proof of

By Handshaking for Faces,

$$\begin{aligned}2|E| &= \sum_{f\in F}\deg(f)\\ &\geqslant 3|F|\end{aligned}$$

where the last inequality holds because each face has degree $\geqslant 3$. Thus,

$$|F|\leqslant\frac{2}{3}|E|$$

LEMMA 7.4.3

If $G = (V, E)$ is a connected planar graph with $|V| \geqslant 3$, then

$$|E| \leqslant 3|V| - 6$$

Proof of

We know that $|V| - |E| + |F| = 2$ and $|F| \leqslant \frac{2}{3}|E|$, where F is the set of faces in some planar drawing of G. Combining, yields

$$2 = |V| - |E| + |F| \leqslant |V| - |E| + \frac{2}{3}|E|$$
$$\implies 2 \leqslant |V| - \frac{1}{3}|E|$$
$$\implies |E| \leqslant 3|V| - 6$$

COROLLARY 7.4.4

The graph K_5 is not planar.

Proof of

K_5 has 5 vertices and 10 edges, but

$$10 \nleqslant 3 \cdot 5 - 6 = 9$$

As we could see, K_5 has too many edges to be planar. If we removed one edge, we could get a planar graph:

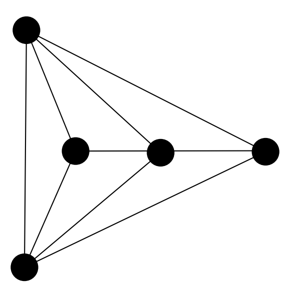


Figure 7.1: $K_5 - e$

PROPOSITION 7.4.5

If G is connected graph on $\geqslant 3$ vertices with $|E| \geqslant 3|V| - 5$, then G is non-planar.

Proof of

Contrapositive of Lemma 7.4.3.

We can't use this proposition to prove $K_{3,3}$ is non-planar.

Recall The boundary of any face contains a cycle (in a plane drawing of a graph that is not a tree).

COROLLARY 7.4.6

In a plane drawing of a graph $G = (V, E)$ with $|V| \geqslant 3$ and set of faces F, such that G has no 3-cycle every face has degree $\geqslant 4$.

COROLLARY 7.4.7

In a graph G as above,

$$|F| \leqslant \frac{1}{2}|E|$$

Proof of

Handshaking with Faces gives,

$$2|E| = \sum_{f \in F} \deg(f) \geqslant 4|F|$$

COROLLARY 7.4.8

If $G = (V, E)$ is planar and connected, has $\geqslant 3$ vertices, and has no 3-cycle, then

$$|E| \leqslant 2|V| - 4$$

Proof of

$$2 = |V| - |E| + |F| \leqslant |V| - |E| + \frac{1}{2}|E|$$

$$2 \leqslant |V| - \frac{1}{2}|E| \implies |E| \leqslant 2|V| - 4$$

COROLLARY 7.4.9

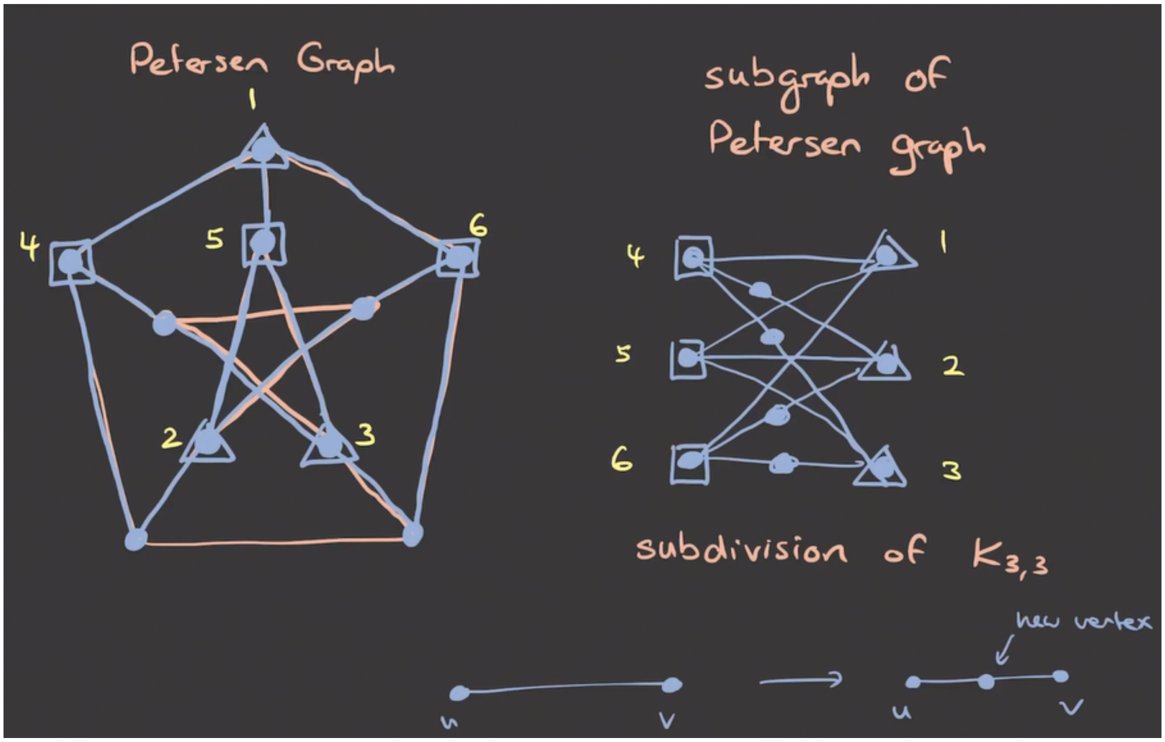
$K_{3,3}$ is non-planar.

Proof of

$K_{3,3}$ has $|V| = 6$ and $|E| = 9$, but

$$9 \nleqslant 2 \cdot 6 - 4 = 8$$

K_5 and $K_{3,3}$ are non-planar, so are all their super graphs.



DEFINITION 7.4.10

If uv is an edge in a graph G, the graph G' is obtained by ***subdividing*** the edge uv has vertex set

$$V(G) \cup \{x\}$$

where x is a new vertex, and edge set

$$(E(G) \setminus \{uv\}) \cup \{ux, vx\}$$

DEFINITION 7.4.11

A ***subdivision*** of a graph G is any graph obtained from G by repeatedly ($\geqslant 0$) subdividing edges.

PROPOSITION 7.4.12

If H is a non-planar graph, then every subdivision of H is also non-planar.

Proof of

Any planar drawing of a subdivision of H would give rise to a plane drawing of H.

COROLLARY 7.4.13

If H is a non-planar graph, and G is a graph having a subdivision of H as a subgraph, then G is non-planar.

7.5 Kuratowski's Theorem

THEOREM 7.5.1: Kuratowski's Theorem

A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.

Proof of

Beyond the scope of this course. However, the proof is covered in CO 342.

COROLLARY 7.5.2

If G is non-planar, then G must contain (as a subgraph) a subdivision of either K_5 or $K_{3,3}$.

Proof of

Contrapositive of Theorem 7.5.1.

THEOREM 7.5.3

For any topological surface, there is a finite list of graphs that behave like $K_{3,3}$ and K_5 do for the plane.

2020-03-30

7.6 Colouring and Planar Graphs

DEFINITION 7.6.1

Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}_{\geqslant 1}$. A k-***colouring*** of G is an assignment of 'colours' from the set $\{1, \ldots, k\}$ to the vertices of G so that the adjacent vertices receive different colours.

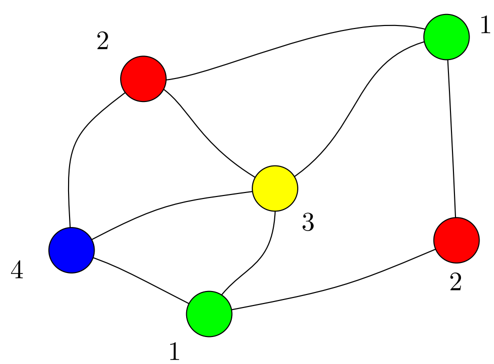


Figure 7.2: 4-colouring of a graph.

DEFINITION 7.6.2

The ***chromatic number***, denoted $\chi(G)$, of a graph G is the minimum k such that G has a k-colouring.

Four-colour problem: Is it true that every planar graph has chromatic number $\leqslant 4$?

Answer: Yes. Appel and Hacken (1977). The proof was one of the first done with a computer.

LEMMA 7.6.3

Every non-empty planar graph has a vertex of degree $\leqslant 5$.

Proof of

The average degree of a vertex v of $G = (V, E)$ is

$$\begin{aligned} \frac{1}{|V|} \sum_{v \in V(G)} \deg(v) &= \frac{2|E|}{|V|} && \text{Handshaking Lemma} \\ &\leqslant \frac{2(3|V| - 6)}{|V|} && \text{by Lemma 7.4.3} \\ &= 6 - \frac{12}{|V|} \end{aligned}$$

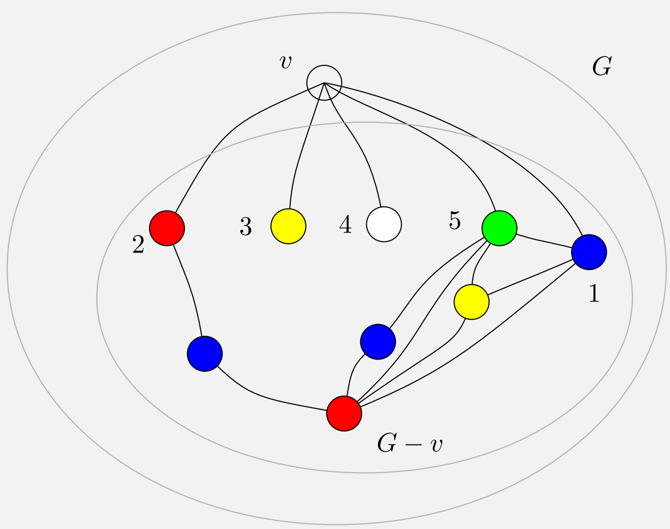
so some vertex has degree $\leqslant 6 - \frac{12}{|V|}$ and therefore $\leqslant 5$.

THEOREM 7.6.4: 6-colour Theorem

Every planar graph is 6*-colourable.*

Proof of

We prove this by induction on the number of vertices. Clearly, every graph with 1 vertex is 6-colourable. Let G be a planar graph on n vertices, and suppose inductively that every planar graph on $n-1$ vertices is 6-colourable. Let v be a vertex of G whose degree is $\leqslant 5$.



Inductively, a planar graph $G - v$ is 6-colourable. Since v has at most 5 neighbours, there is a colour in $\{1, 2, 3, 4, 5, 6\}$ not assigned to any neighbour of v in G. Assigning this colour to v gives a 6-colouring of G.

THEOREM 7.6.5: Five-colour Theorem

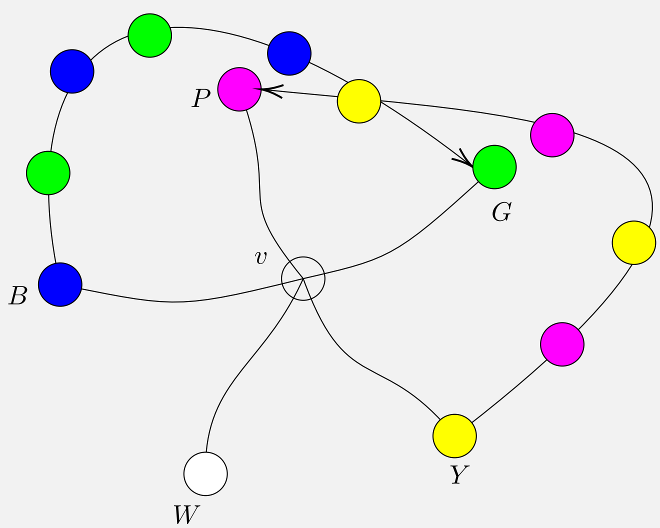
Every planar graph is 5*-colourable.*

Proof of

Induction on $|V(G)|$. Let G be a graph on $n \geqslant 1$ vertices and suppose the theorem holds for every planar graph $n-1$ vertices. Let v be a vertex of degree $\leqslant 5$ in G. Inductively, $G - v$ has a 5-colouring. If at most 4 colours are assigned to neighbours of v, then we can extend the colouring of $G - v$ to a colouring of G by choosing a colour for v not appearing on neighbours of v. Otherwise, there are $\geqslant 5$ colours appearing on neighbours of v. Since v has $\leqslant 5$ neighbours, this means every neighbour of v is assigned a different colour. Let

$$B,\ P,\ G,\ Y,\ W$$

be the colours occurring on the neighbours of v as they appear in clockwise order around v.



Let G_{GB} be the subgraph of $G - v$ induced by the green and blue vertices. Define G_{PY} similarly. By the planarity of G, either the G, B neighbours of v are disconnected in G_{GB}, or the P, Y neighbours of v are disconnected in G_{PY}. Suppose by symmetry the first case holds.
Let u, w be the B, G neighbours of v so u, w are disconnected in G_{GB}. Let C be the component of G_{GB} containing u. Now, form a new 5-colouring of $G - v$ by switching the colour of every vertex of C from G to B or vice versa. This gives a 5-colouring of $G - v$ in which both u and w are G (because $w \notin C$). Now, we can colour v blue to get a 5-colouring of G.

Chapter 8

Matchings

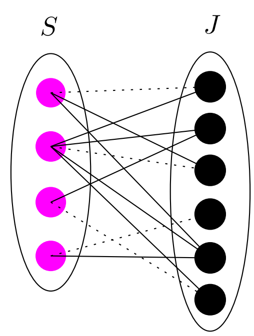
2020-03-01

8.1 Matching

Given a set S of students and a set J of co-op jobs, how many positions can we fill given that

- each student in S has some set of jobs they are willing to do, and
- each job in J has some set of students that can do it

We can model this situation with a graph (where the set of dotted lines is a possible matching)



the vertices are $S \cup J$, and we include an edge between student s and job j if and only if they are 'compatible'.

DEFINITION 8.1.1

A ***matching*** in a graph G is a set M of edges of G so that no two edges in M have an end in common.

Asking for as many job assignments as possible is equivalent to asking for a largest possible matching in the graph.

DEFINITION 8.1.2

Let $G = (V, E)$ be a graph, and M be a matching of G

- A vertex v of G is ***saturated*** by M if some edge in M has v as an end. Otherwise it is ***unsaturated*** (exposed). Note: A matching of size k has $2k$ saturated vertices.
- M is a ***maximum matching*** of G if G has no matching larger than M.
- M is a ***maximal matching*** if M is contained in no larger matching of G. Note: Maximum $\implies$ Maximal, but the converse is not true.

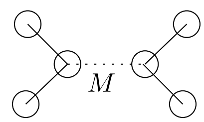


Figure 8.1: Maximal, but not maximum matching.

DEFINITION 8.1.3

Let P be a path in a graph $G = (V, E)$ and M be a matching of G. Let $e_1, e_2, \ldots, e_k$ be the edges of P occurring in order. If $M \cap E(P)$ is either $\{e_1, e_3, e_5, \ldots\}$ or $\{e_2, e_4, e_6, \ldots\}$ then P is an ***M-alternating path***. That is, the edges in P alternate between being matching and non-matching edges. P is an ***M-augmenting path*** if $M \cap E(P) = \{e_2, e_4, e_6, \ldots\}$, and also both ends of P are unsaturated vertices.

We can use an M-augmenting path in a graph G to make a matching M' of G that is larger than M.

PROPOSITION 8.1.4

If M is a matching in G and P is an M-augmenting path, then M is not a maximum matching of G.

Proof of

Replace M with

$$(M \setminus (M \cap E(P))) \cup (E(P) \setminus M)$$

to get a larger matching than M.

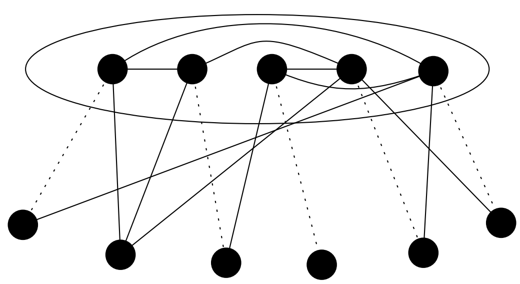


Figure 8.2: Vertex Cover

Clearly, it is not possible to have a matching of size 6 here. The matching above covers all edges of the graph.

8.2 Covers

DEFINITION 8.2.1

A (vertex) ***cover*** of a graph G is a set $C \subseteq V(G)$ such that every edge of G has at least one end in C.

PROPOSITION 8.2.2

If C is a cover of G and M is a matching of G, then

$$|M| \leqslant |C|$$

Proof of

We use the pigeonhole principle. Each edge in M has an end in C by the definition of a cover. Since M is a matching, these ends are distinct. So

$$|C| \geqslant |M|$$

PROPOSITION 8.2.3

If C is a cover of G and M is a matching of G with $|M| = |C|$, then M is a maximum matching of G, and C is a minimum (smallest possible) cover of G.

Proof of

Let M_0 be a maximum matching, and C_0 be a minimum cover. Then,

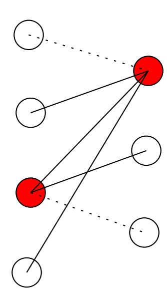
$$\begin{aligned} |M| &\leqslant |M_0| && M_0 \text{ is a maximum matching} \\ &\leqslant |C_0| && \text{by Proposition 8.2.2} \\ &\leqslant |C| && C_0 \text{ is a minimum cover} \\ &= |M| && \text{by assumption} \end{aligned} \tag{8.1}$$

Equality holds throughout, so $|M| = |M_0|$, which means that M is a maximum matching and $|C| = |C_0|$ means that C is a minimum cover.

†Note: Usually, $\nu(G)$ = size of a maximum matching of G, and $\tau(G)$ = size of a minimum cover of G.

Proposition 8.2.2 states that, for any G,

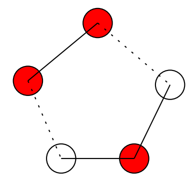
$$|\text{maximum matching of } G| \leqslant |\text{minimum cover of } G|$$



Deleting the red vertices $\implies$ $|C| = 2$.

$$|\text{maximum matching of G}| = 2 = |\text{minimum cover of } G|$$

However, this is not always true as seen below.



$$|\text{maximum matching of G}| = 2 \neq 3 = |\text{minimum cover of } G|$$

2020-03-03

8.3 König's Theorem

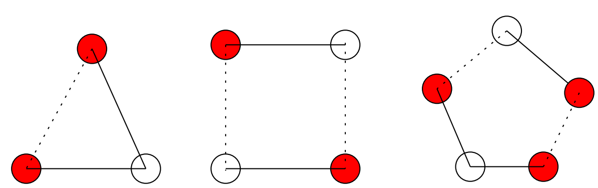
THEOREM 8.3.1: König's Theorem

If G is a bipartite graph, then

$$|\textit{maximum matching of } G| = |\textit{minimum cover of } G|$$

Let's first look at few more examples of matchings and covers since they will form the base case of Theorem 8.3.1.

Cycles: Let C_n be the n-cycle.



It is easy to see that

$$|\text{maximum matching of } C_n| = \left\lfloor \frac{n}{2} \right\rfloor$$
$$\left(= \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases} \right)$$

For even cycles, (every other vertex) is a cover of size $\frac{n}{2}$, so is a minimum cover by Proposition 8.2.3. For odd cycles, the minimum cover has size $\lceil \frac{n}{2} \rceil$ because $\lfloor \frac{n}{2} \rfloor$ is not enough since every vertex has degree 2.

So, for C_n,

$$|\text{maximum matching of } C_n| = \left\lfloor \frac{n}{2} \right\rfloor$$

$$|\text{minimum cover of } C_n| = \left\lceil \frac{n}{2} \right\rceil$$

Therefore, $|\text{maximum matching}| = |\text{minimum cover}|$ if and only if n is even; that is, C_n is bipartite.

Paths: Let P_n be a path on n vertices. We take every other edge to be in the maximum matching.

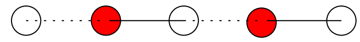


Figure 8.3: P_5

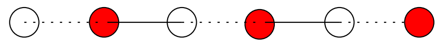


Figure 8.4: P_6

It is easy to check that

$$|\text{maximum matching of } P_n| = \left\lfloor \frac{n}{2} \right\rfloor$$

$$|\text{minimum cover of } P_n| = \left\lfloor \frac{n}{2} \right\rfloor$$

Thus,

$$|\text{maximum matching of } P_n| = |\text{minimum cover of } P_n|$$

It follows that, if G is a bipartite graph where every component is a path or cycle, then

$$|\text{maximum matching of } G| = |\text{minimum cover of } G|$$

Proof of

A maximum matching of G consists of a maximum matching for each component. Same for the minimum cover. Since we know $|\text{maximum matching}| = |\text{minimum cover}|$ for each component, the same is true for G.

PROPOSITION 8.3.2

Let G be a graph with no vertices of degree 3 or more. Then every component of G is a path or cycle.

Proof of

Let C be a component of G. Let P be a longest path in C. Let u and v be the ends of P. Since every vertex of P has degree $\geqslant 2$, the only possible edge in C that is not in P is the edge from u to v. Therefore, either $P = C$, or C is a cycle. So every component is a path or cycle.

PROPOSITION 8.3.3

If M is a matching of G and C is a cover of G with $|M| = |C|$, then every vertex in C is the end of an edge in M.

Proof of

Since C is a cover, every edge in M has an end in C; since M is a matching, these ends are all different, so there is nothing else in C.

We can now prove Theorem 8.3.1.

THEOREM 8.3.4: König's Theorem (1931)

If G is a bipartite graph, then

$$|\textit{maximum matching of } G| = |\textit{minimum cover of } G|$$

The proof in course notes is longer since it is used for the algorithm: a maximum matching in a bipartite graph, that is usually within this course, but due to the lack of time, we will not present it here, and hence the proof presented will be different.

Proof of

Rizzi (1999). We use strong induction. The theorem is obvious for graphs with no edges; let G be a bipartite graph with $m > 0$ edges, and suppose that the theorem holds for every bipartite graph with $< m$ edges. Let k be the size of a maximum matching. If every vertex of G has degree $\leqslant 2$, then every component is a path or cycle, so König's Theorem holds. Let u be a vertex of G of degree $\geqslant 3$, and let v be a neighbour of u.
Case 1: The graph $G - v$ has no matching of size k. If this is true, the graph $G - v$ has a maximum matching if size $\leqslant k - 1$. Since $G - v$ is bipartite and has fewer edges than G, the inductive hypothesis implies that $G - v$ has a cover C_1 of size $\leqslant k - 1$. So $C_1 \cup \{v\}$ is a cover of G of size $\leqslant k$. Since k is the size of a matching of G, we have $|C_1 \cup \{v\}| \geqslant k$, so $C_1 \cup \{v\}$ is a cover of the same size as a matching of G, so by Proposition 8.3.3, we have

$$|\text{maximum matching}| = |\text{minimum cover}|$$

Case 2: The graph $G - v$ does have a matching of size k. Let M be such a matching; that is, $|M| = k$ and M does not saturate v. Let f be an edge not in M, that is incident with u but not v. Since $G - f$ is bipartite and has fewer edges than G, and has a matching M of size k, inductively $G - f$ has a cover C_2 of size k. Since $|C_2| = |M|$ every vertex in C_2 is saturated by M by Proposition 8.3.3, so $v \notin C_2$. Since C_2 contains $\geqslant 1$ end of the edge uv, we have $u \in C_2$. So, C_2 contains an end of f, and an end of every edge in $G - f$. So C_2 is a cover of G of size $k = |\text{maximum matching}|$. So C_2 is a minimum cover; thus

$$|\text{maximum matching}| = |\text{minimum cover}|$$

2020-03-06

8.4 Applications of König's Theorem

The matching problem for bipartite graphs is in Co-NP.

Q: Given a bipartite graph G with bipartition (A, B), when does G have a matching saturating A?

If $|A| = |B|$, then this is asking for a ***perfect matching*** of G, which is a matching that saturates every vertex. Note that, if some vertex in A has no neighbours in B, there is no matching saturating A. The same is true if there are two vertices u, v in A such that u and v both only have one neighbour w in B.

For each set $A' \subseteq A$, let $N(A')$ denote the set of vertices B having a neighbour in A'.

PROPOSITION 8.4.1

If $A' \subseteq A$ is a set for which $|N(A')| < |A'|$, then G has no matching saturating A.

Proof of

A matching saturating A must matching each vertex in A' to a distinct vertex in $N(A')$, but $N(A')$ is too small for there to exist this many distinct vertices.

THEOREM 8.4.2: Hall's Theorem

Let G be a bipartite graph with bipartition (A, B). Then exactly one of the following holds.
(1) G has a matching saturating A
(2) There is a set $A' \subseteq A$ such that $|N(A')| < |A'|$
In other words, G has a matching saturating A if and only if $|N(A')| \geqslant |A'|$ for all $A' \subseteq A$.

Proof of

We have seen that $(2) \implies (1)$ doesn't hold. $(2) \implies \neg(1)$. It is now enough to show that $\neg(2) \implies (2)$. Suppose that (1) doesn't hold; that is, G has no matching saturating A. Let M be a maximum matching of G, so $|M| < |A|$. By König's Theorem, G has a cover C of size $|M| < |A|$. Let $C_A = C \cap A$ and $C_B = C \cap B$. Let $A' = A \setminus C_A$. Since C is a cover, we know that $N(A') \subseteq C_B$. Now

$$\begin{aligned}
|N(A')| &\leqslant |C_B| \\
&\leqslant |C| - |C_A| && \text{since } C = C_A \cup C_B \\
&= |M| - |C_A| && \text{by Theorem 8.3.1} \\
&< |A| - |C_A| && \text{by assumption} \\
&= |A'|
\end{aligned}$$

so (2) holds because $|N(A')| < |A'|$.

REMARK 8.4.3: A

signment 10 can now be completed up to this point.

<u>**Chess**</u>

<u>Problem</u>: Given a square $n \times n$ array, find a set X of n cells in the array, with no two in the same row or column. Let A be the set of rows, B be the set of columns.

PROPOSITION 8.4.4

If there is a set $A' \subseteq A$ and a set $B' \subseteq B$ such that $|A'| + |B'| > n$ and every cell in $A' \times B'$ is 'blocked', then placing n non-attacking rooks is impossible.

Proof of

$|A'|$ rooks must be placed in $\leqslant n - |B'| < |A'|$ different columns.

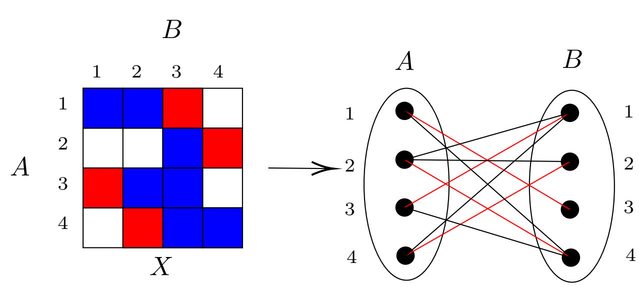


Figure 8.5: Chess to Bipartition

Given a set $X \subseteq A \times B$ of 'allowable' squares in an $A \times B$ array, we can form a bipartite graph $G(X)$ with partition (A, B) where $a \in A$ is adjacent to $b \in B$ if and only if the ab-cell is in X.

PROPOSITION 8.4.5

If $|A| = |B| = n$, then we can place n non-attacking rooks in the array if and only if G has a perfect matching.

By Hall's Theorem, there is a matching saturating A if and only if there is no set $A' \subseteq A$ for which $|N(A')| < |A'|$.

If $|N(A')| < |A'|$ in $G(X)$, then this means that every cell in $A' \times (B \setminus N(A'))$ is forbidden (not in X). But $|A'| + |B \setminus N(A')| = |A'| + n - |N(A')| > n$.

PROPOSITION 8.4.6

Given a square $A \times B$ array and a set F of forbidden squares ($F \subseteq A \times B$), either

- *we can place $n = |A| = |B|$ non-attacking rooks while avoiding all squares in F, or*
- *there is a set $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| + |B'| > n$ and every square in $A' \times B'$ is forbidden.*

Linear Algebra

$$\begin{pmatrix} \star & \star & 0 & 0 & \star & 0 & \star \\ 0 & 0 & \star & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & \star \\ 0 & \star & \star & 0 & 0 & \star & 0 \\ 0 & \star & 0 & 0 & \star & 0 & 0 \\ \star & 0 & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & \star \end{pmatrix}_{7\times 7}$$

Question: Given a square '$\star$-0-matrix' when can we find non-zero entries for the $\star$'s such that the matrix is non-singular.

Example: Row of 0's forces singular.

PROPOSITION 8.4.7

Given a square '$\star$-0-matrix' with rows A and columns B, either

- *we can find non-zero entries for the $\star$'s such that the determinant is non-zero, or*
- *there is a set $A' \subseteq B$ and $B' \subseteq B$ such that $|A'| + |B'| > n$ and every entries in $A' \times B'$ is zero, so the determinant is forced to be 0.*

Proof of

Apply Hall's Theorem.

Sequel Courses:

- First half of MATH 239: CO 330 (Algebraic Enumeration)
- Second half of MATH 239: CO 342 (Graph Theory)

Chapter 9

Tutorials

9.1 Tutorial 1

Problem 1. Give a combinatorial proof that the number of subsets of $\{1, \ldots, n\}$ with even cardinality is the same as the number with odd cardinality.

Solution.

Let X denote any subset of $\{1, \ldots, n\}$. The corresponding set Y will be:

$$Y = \begin{cases} X \cup \{1\}, & \text{if } 1 \notin X \\ X \setminus \{1\}, & \text{if } 1 \in X \end{cases}$$

Problem 2. Let n be a positive integer. Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} i \binom{n}{i} = n2^{n-1}.$$

Solution.

Suppose we have a group of n people.

RHS: Choose a committee of size i, then choose one of the i committee members to be a leader. There are $\binom{n}{i}$ ways to pick the members of the committee and i ways to choose the leader, which yields $\sum\limits_{i=0}^{n} i\binom{n}{i}$.

LHS: Pick a leader, then from the remaining $(n-1)$ people we choose them to either be in or out of the committee. There are n ways to pick the leader and 2^{n-1} ways to pick the remaining committee members.

Thus, since we are counting the same object twice in two different ways, we have that

$$\sum_{i=0}^{n} i \binom{n}{i} = n2^{n-1}.$$

Problem 3. For any integers n, k, r where $n \geqslant k \geqslant r \geqslant 0$, give a combinatorial proof of the following identity.

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

Solution.

Suppose we have a group of n people, with a k-person committee and a r-person subcommittee.

RHS: Choose the committee in $\binom{n}{k}$ ways, then choose the subcommittee from the committee in $\binom{k}{r}$ ways, which yields $\binom{n}{k}\binom{k}{r}$.

LHS: Choose the r subcommittee members in $\binom{n}{r}$ ways, then fill in the remaining $(k-r)$ committee members from the remaining $(n-r)$ people, which yields $\binom{n}{r}\binom{n-r}{k-r}$.

Thus, since we are counting the same object twice in two different ways, we have that

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

Problem 4. Let $n \geqslant 5$ be an integer. Give a combinatorial proof of the following identity

$$\sum_{k=5}^{n}\binom{k-1}{4} = \sum_{m=3}^{n-2}\binom{m-1}{2}\binom{n-m}{2}.$$

(Hint: Both sides are equal to $\binom{n}{5}$.)

Solution.

Too hard for my poor soul.

9.2 Tutorial 2

Problem 1. Use the negative binomial theorem and substitutions to give a formula for the coefficient of x^n in $(1-3x)^{-1} + 2(1-2x)^{-2}$.

Solution. Recall:

THEOREM 9.2.1: Negative Binomial Theorem

Let $m, k \in \mathbb{Z}_{\geqslant 0}$, then

$$(1-x)^{-k} = \sum_{m \geqslant 0}\binom{m+k-1}{k-1}x^m$$

$$(1-3x)^{-1} = \sum_{m\geqslant 0}\binom{m+1-1}{1-1}(3x)^m = \sum_{m\geqslant 0}3^m x^m$$

$$\implies [x^n](1-3x)^{-1} = 3^n$$

$$2(1-2x)^{-2} = 2\sum_{m\geqslant 0}\binom{m+2-1}{2-1}(2x)^m = 2\sum_{m\geqslant 0}(m+1)2^m x^m$$

$$\implies [x^n]2(1-2x)^{-2} = 2(n+1)2^n = (n+1)2^{n+1}$$

Combining,

$$[x^n]\left[(1-3x)^{-1} + 2(1-2x)^{-2}\right] = 3^n + (n+1)2^{n+1}$$

Problem 2. Let $F(x) = x + x^2 + \cdots$ and let $G(x) = 1 + 3x + 2x^2$. Compute the coefficient of x^n in $G(F(x))$.

Solution.

$$G(F(x)) = 1 + 3(x + x^2 + \cdots) + 2(x + x^2 + \cdots)^2$$

$$\begin{aligned} 3(x + x^2 + \cdots) &= 3x(1 + x + x^2 + \cdots) \\ &= \frac{3x}{1-x} \\ &= 3x \sum_{m \geqslant 0} x^m \end{aligned}$$

$$\begin{aligned} 2(x + x^2 + \cdots)^2 &= 2\left[x(1 + x + x^2 + \cdots)\right]^2 \\ &= 2\left(\frac{x}{1-x}\right)^2 \\ &= 2x^2(1-x)^{-2} \\ &= 2x^2 \sum_{m \geqslant 0} \binom{m+2-1}{2-1} x^m \\ &= 2x^2 \sum_{m \geqslant 0} (m+1)x^m \end{aligned}$$

Computing coefficients,

$$[x^n]1 = \begin{cases} 0, & \text{if } n \geqslant 1 \\ 1, & \text{if } n = 0 \end{cases}$$

$$\begin{aligned} [x^n]3(x + x^2 + \cdots) &= [x^n]3x \sum_{m \geqslant 0} x^m \\ &= [x^{n-1}]3 \sum_{m \geqslant 0} x^m \\ &= \begin{cases} 3, & \text{if } n \geqslant 1 \\ 0, & \text{if } n = 0 \end{cases} \end{aligned}$$

$$\begin{aligned} [x^n]2(x + x^2 + \cdots)^2 &= [x^n]2x^2 \sum_{m \geqslant 0} (m+1)x^m \\ &= [x^{n-2}]2 \sum_{m \geqslant 0} (m+1)x^m \\ &= \begin{cases} 2n-2, & \text{if } n \geqslant 2 \\ 0, & \text{if } n \in \{0, 1\} \end{cases} \end{aligned}$$

Thus,

$$[x^n]G(F(x)) = \begin{cases} 2n+1, & \text{if } n \geq 2 \\ 3, & \text{if } n = 1 \\ 1, & \text{if } n = 0 \end{cases}$$

Problem 3. Show that if $F(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then

$$F(x)(1-x) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \cdots$$

and

$$F(x)(1-x)^{-1} = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = a_0 + a_1 + \cdots + a_n$.

Solution.

Part 1.

$$\begin{aligned} F(x)(1-x) &= (a_0 + a_1 x + a_2 x^2 + \cdots)(1-x) \\ &= a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 - a_2 x^3 - a_3 x^4 + \cdots \end{aligned}$$

as required.

Part 2.

We know that $(1-x)^{-1} = 1 + x + x^2 + \cdots$, thus

$$\begin{aligned} F(x)(1-x)^{-1} &= F(x) + xF(x) + x^2F(x) + \cdots \\ &= \sum_{i \geqslant 0} x^i F(x) \\ &= \sum_{i \geqslant 0} [x^{n-i}]F(x) \\ &= \sum_{i=0}^{n} a_{n-i} \text{ for } 0 \leq k = n - i \leq n \\ &= \sum_{k=0}^{n} a_k \end{aligned}$$

where

$$[x^{n-i}]F(x) = \begin{cases} a_{n-i}, & \text{if } i \leqslant n \\ 0, & \text{if } i > n \end{cases}$$

Problem 4. Show that for $k \geqslant 1$ and $n \geqslant 1$, we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} = 0,$$

where we interpret $\binom{j}{i} = 0$ when $j < i$. Hint: Look at $1 = (1-x)^k(1-x)^{-k}$ and compute the coefficient of x^n in both sides.

Solution.

$$[x^n]1 = \begin{cases} 0, & \text{if } n \geqslant 1 \\ 1, & \text{if } n = 0 \end{cases}$$

$$(1-x)^{-k}(1-x)^k = \sum_{j \geqslant 0} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{j+k-1}{k-1} x^{j+i} \qquad \text{Negative Bin. and Bin. Theorem}$$

Let $j + i = n \iff j = n - i$ to compute $[x^n](1-x)^{-k}(1-x)^k$, and we get

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} = 0$$

for $k \geqslant 1$ and $n \geqslant 1$ as desired.

9.3 Tutorial 3

Problem 1. Consider the set of non-negative integers $\mathbb{N}_0$, but with a non-standard weight function

$$w(a) = \begin{cases} \frac{3}{2}a + 1, \text{ if } a \text{ is even,} \\ 2(a+1), \text{ if } a \text{ is odd.} \end{cases}$$

Find the generating series for $\mathbb{N}_0$ with respect to this weight function and express it as a simplified rational expression.

Solution.

$\mathbb{N}_0 = \mathbb{N}_{\text{even}} \cup \mathbb{N}_{\text{odd}}$

$$\begin{aligned}
\Phi_{\mathbb{N}_0}(x) &= \Phi_{\mathbb{N}_{\text{even}}}(x) + \Phi_{\mathbb{N}_{\text{odd}}}(x) \qquad \text{Sum Lemma} \\
&= \sum_{a \text{ even}} x^{3/2a+1} + \sum_{a \text{ odd}} x^{2(a+1)} \\
&= x \sum_{a \text{ even}} x^{3/2a} + x^2 \sum_{a \text{ odd}} x^{2a} \\
&= x \sum_{i \geq 0} (x^{3/2})^{2i} + x^2 \sum_{i \geq 0} (x^2)^{2i+1} \\
&= x \sum_{i \geq 0} (x^3)^i + x^4 \sum_{i \geq 0} (x^4)^i \\
&= \frac{x}{1-x^3} + \frac{x^4}{1-x^4}
\end{aligned}$$

Problem 2. Let m, n be positive integers and α, β positive real numbers. Find the generating series for the cartesian product

$$\{1, \ldots, m\} \times \{1, \ldots, n\}$$

with respect to the weight function

$$w(a,b) = \alpha a + \beta b$$

and express it as a simplified rational expression.

Solution.

THEOREM 9.3.1: Partial Geometric Series

$$\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$$

$$\begin{aligned}
\Phi_{A \times B}(x) &= \sum_{(a,b) \in A \times B} x^{\alpha a + \beta B} \\
&= \sum_{a \in A} x^{\alpha a} \sum_{b \in B} x^{\beta b} && \text{Product Lemma} \\
&= x^{\alpha+\beta} \sum_{i=1}^{m} (x^a)^i \sum_{j=1}^{n} (x^b)^j \\
&= x^{\alpha+\beta} \left(\frac{x^a(1-x^{am})}{1-x^a} \right) \left(\frac{x^b(1-x^{bm})}{1-x^b} \right) && \text{Partial Geometric Series} \\
&= x^{\alpha+\beta+a+b} \frac{(1-x^{am})(1-x^{bm})}{(1-x^a)(1-x^b)}
\end{aligned}$$

Problem 3. Let a, b, n, k be positive integers with $a \leq b$ and $k \leq n$. How many compositions of n with k parts are there in which all parts are elements of $\{a, \ldots, b\}$? Expressing the result as a finite sum $\sum\limits_{i=0}^{k} s_i$ is sufficient.

Rough.

Start with a small example $a = 2,\ b = 4,\ n = 9,\ k = 3$.

Let $C = \{$compositions with 3 parts where each part is 2, 3, or 4$\}$. We want $[x^9]\Phi_C(x)$.

Let $P = \{$parts of value 2, 3, or 4$\}$. $C = P \times P \times P$ (3 parts). Thus, $\Phi_C(x) = (\Phi_P(x))^3$.

$$(\Phi_P(x))^3 = \left(\sum_{i=2}^{4} x^i\right)^3$$

We want $[x^9](x^2 + x^3 + x^4)^3$.

Solution.

So, in general we have

$$\begin{aligned}\Phi_C(x) &= (\Phi_P(x))^k \\ &= \left(\sum_{i=a}^{b} x^i\right)^k\end{aligned}$$

We want $[x^n]\left(\sum\limits_{i=a}^{b} x^i\right)^k$.

$$\begin{aligned}[x^n]\left(\sum_{i=a}^{b} x^i\right)^k &= [x^n]\,(x^a + \cdots + x^b)^k \\ &= [x^n]\,x^{ak}(1 + \cdots + x^{b-a})^k \\ &= \left[x^{n-ak}\right]\left(\frac{1 - x^{b-a+1}}{1-x}\right)^k \\ &= \left[x^{n-ak}\right]\left(1 + (-x^{b-a+1})\right)^k (1-x)^{-k} \\ &= \left[x^{n-ak}\right]\sum_{i\geq 0}\binom{k}{i}(-1)^i x^{i(b-a+1)} \sum_{j\geq 0}\binom{j+k-1}{k-1}x^j \\ &= \left[x^{n-ak}\right]\sum_{i\geq 0}\sum_{j\geq 0}\binom{k}{i}\binom{j+k-1}{k-1}(-1)^i x^{i(b-a+1)+j}\end{aligned}$$

$i(b - a + 1) + j = n - ak$

$$\sum_{i=0}^{\lfloor\frac{n-ak}{b-a+1}\rfloor}\binom{k}{i}\binom{n - ak - i(b-a+1) + k - 1}{k-1}(-1)^i$$

9.4 Tutorial 4

Problem 1. Let S denote the set of strings of the form $\{1\}^*\{0\}^*\{1\}^*\{0\}^*$. Find the generating function for $\Phi_S(x)$, where the weight of a string is given by its length.

Solution.

$\{1\}^* \left(\{0\}\{0\}^*\{1\}\{1\}^*\right) \{0\}^* \cup \{1\}^*\{0\}^*$

Problem 2. Let $S = \{00, 111\}^*$. Find a formula for $\Phi_S(x)$.

Solution.

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\{00,111\}^*} \\
&= \frac{1}{1 - \Phi_{\{00,111\}}} \\
&= \frac{1}{1 - (x^2 + x^3)}
\end{aligned}$$

Problem 3. Let S be $\{00, 111\}^*$ and let S_n denote the set of strings of length n in S. Give a combinatorial proof that $|S_n| = |S_{n-2}| + |S_{n-3}|$ for $n \geqslant 3$.

Look at $n = 0, \ldots, 6$.

$n = 2 : |S_2| = 1 \to 00$

$n = 3 : |S_3| = 1 \to 111$

$n = 4 : |S_4| = 1 \to 0000$

$n = 5 : |S_5| = 2 \to 00111, 11100$

$n = 6 : |S_6| = 2 \to 111111, 000000$

Solution.

Let $s \in S_n$, $n \geqslant 3$. What could s start with?

Case 1: $00t$, $t \in S_{n-2}$

Case 2: $111t$, $t \in S_{n-3}$

$f : S_n \to S_{n-2} \cup S_{n-3}$, $\forall s \in S_n$, $f(s) = t$ where

$$s = \begin{cases} 00t, \text{ if } s \text{ starts with } 00 \\ 111t, \text{ if } s \text{ starts with } 111 \end{cases}$$

$g : S_{n-2} \cup S_{n-3} \to S_n$

$$g(t) = \begin{cases} 00t, \ t \in S_{n-2} \\ 111t, \ t \in S_{n-3} \end{cases}$$

Explain how $f(g(t)) = t$ and $g(f(s)) = s$.

Problem 4. Explain why $(\{1\}^*\{0\}^*)$ is ambiguous.

Solution.

Ambiguous means there are multiple ways to create a string. So, taking ε works since,

$$\begin{aligned}
\left(\{1\}^0\{0\}^0\right)^x &= (\varepsilon)^x \\
&= \varepsilon
\end{aligned}$$

9.5 Tutorial 5

Problem 1. Let S denote the set of binary strings not containing the string 101 as a substring. Find an unambiguous expression for S, and use it to give a rational expression for $\Phi_S(x)$, weighted by length.

Solution.

$\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$

$S = \{0\}^*(\{1\}\{1\}^*\{00\}\{0\}^*)^*\{1\}^*\{\varepsilon, 10\}$

$T = \{$binary strings containing exactly one copy of 101 as a suffix$\}$

$$\begin{gathered}
\{\varepsilon\} \cup S\{0,1\} = S \cup T \\
S\{101\} = T \cup T\{01\} \\
1 + \Phi_S(x)2x = \Phi_S(x) + \Phi_T(x) \\
\Phi_S(x)x^3 = \Phi_T(x) + \Phi_T(x)x^2 \\
\implies 1 + \Phi_S(x)2x - \Phi_S(x) = \Phi_T(x)
\end{gathered}$$

substituting,

$$\begin{aligned}
\Phi_S(x)x^3 &= 1 + \Phi_S(x)2x - \Phi_S(x) + x^2 + \Phi_S(x)2x^3 - \Phi_S(x)x^2 \\
\implies \Phi_S(x)x^3 &= 1 + \Phi_S(x)2x - \Phi_S(x) + \Phi_S(x)2x^3 - \Phi_S(x)x^2 + x^2 \\
\implies \Phi_S(x)x^3 - \Phi_S(x)2x + \Phi_S(x) - \Phi_S(x)2x^3 + \Phi_S(x)x^2 &= 1 + x^2 \\
\implies -\Phi_S(x)x^3 - \Phi_S(x)2x + \Phi_S(x) + \Phi_S(x)x^2 &= 1 + x^2 \\
\implies \Phi_S(x)(-x^3 - 2x + 1 + x^2) &= 1 + x^2 \\
\implies \Phi_S(x) &= \frac{1 + x^2}{-x^3 + x^2 - 2x + 1}
\end{aligned}$$

where the algebra has been verified with WolframAlpha.

Problem 2. Let S be the set of binary strings with an odd number of blocks. Find an unambiguous recursive decomposition for S, and use it to find a rational expression for $\Phi_S(x)$, weighted by length.

$X = \{$binary strings with odd number of blocks, beginning with $1\}$

$Y = \{$binary strings with odd number of blocks, beginning with $0\}$

$S \cup T = T\{0,1\} \cup S\{0,1\} \cup \{\varepsilon\}$

$S = X \cup Y$

$X = \{1\}\{1\}^*(\{0\}\{0\}^*X \cup \{\varepsilon\}) \to \Phi_X(x) = \frac{x}{1-x}\left(\frac{x}{1-x}\Phi_X(x) + 1\right)$

$Y = \{0\}\{0\}^*(\{1\}\{1\}^*Y \cup \{\varepsilon\})$

$$\Phi_X(x) = \frac{x^2}{(1-x)^2}\Phi_X(x) + \frac{x}{1-x}$$

$$\begin{aligned}
\implies \Phi_X(x) &= \frac{\frac{x}{1-x}}{1 - \frac{x^2}{(1-x)^2}} \\
&= \frac{x(1-x)}{(1-x)^2 - x^2} \\
&= \frac{x - x^2}{1 - 2x}
\end{aligned}$$

$$\Phi_S(x) = \frac{2x - 2x^2}{1 - 2x}$$

Problem 3. Let k and ℓ be non negative, and S be the set of binary strings in which no block of zeros has length greater than k and no blocks of ones has length greater than ℓ. Find an unambiguous recursive decomposition for S, and use it to find a rational expression for $\Phi_S(x)$, weighted by length.

$T = \{0, 00, \dots, 0^k\}$, $U = \{1, 11, \dots, 1^\ell\}$

$(T \cup \{\varepsilon\})(UT)^*(U \cup \{\varepsilon\})$

$$\Phi_T(x) = \frac{x(1 - x^k)}{1 - x}$$

$$\Phi_U(x) = \frac{x(1 - x^\ell)}{1 - x}$$

9.6 Tutorial 6

Problem 1. Let $n \geqslant 0$. Use partial fractions to compute the coefficient of x^n in

$$\frac{x(x-1)}{x^3 + 6x^2 + 11x + 6}$$

Solution.

$$\begin{aligned}
\frac{x(x-1)}{x^3 + 6x^2 + 11x + 6} &= \frac{x^2 - x}{(x+1)(x+2)(x+3)} \\
&= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\
&= A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) \\
&= A(x^2 + 5x + 6) + B(x^2 + 4x + 3) + C(x^2 + 3x + 2) \\
&= (6A + 3B + 2C) + (5A + 4B + 3C)x + (A + B + C)x^2
\end{aligned}$$

Equating coefficients, gives three equations and three unknowns:

$$6A + 3B + 2C = 0$$

$$5A + 4B + 3C = -1$$

$$A + B + C = 1$$

$$\left[\begin{array}{ccc|c} 6 & 3 & 2 & 0 \\ 5 & 4 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Thus, we have $A = 1$, $B = -6$, $C = 6$.

$$\begin{aligned}
\frac{1}{x+1} + \frac{-6}{x+2} + \frac{6}{x+3} &= (-1)^n - 6[x^n]\frac{1}{x+2} + 6[x^n]\frac{1}{x+3} \\
&= (-1)^n - 6(-2)^n + 6(-3)^n
\end{aligned}$$

Problem 2. Solve the following recurrences:

(a) $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ for $n \geqslant 3$ with initial conditions $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$.

(b) $a_n - za_{n-2} = 0$ for $n \geqslant 2$, $z \in \mathbb{R}$, with initial conditions $a_0 = 1$ and $a_1 = 2z$.

Solution.

(a) The polynomial is $x^3 - 3x^2 + 3x - 1 = 0 \implies (x-1)^3 = 0$. The multiplicity is 3 so the degree is at most 2. $a_n = 1^n(An^2 + Bn + C)$

$$a_0 = C = 0$$

$$a_1 = A + B + C = 1$$

$$a_2 = 4A + 2B + C = 2$$

$A = 0,\ B = 1,\ C = 0$. Substituting gives,

$$a_n = n$$

(b) The polynomial is $x^2 - z = 0 \implies (x + \sqrt{z})(x - \sqrt{z}) = 0$. The roots are $x = \sqrt{z}$ and $x = -\sqrt{z}$ both with multiplicity of 1, so the degree is at most 0. $a_n = (\sqrt{z})^n A + (-\sqrt{z})^n B$

$$a_0 = A + B = 1$$

$$a_1 = \sqrt{z}A - \sqrt{z}B = 2z$$

Solving, gives $A = {}^1\!/_2 + \sqrt{z}$ and $B = {}^1\!/_2 - \sqrt{z}$. Substituting gives,

$$a_n = \sqrt{z}\left(\frac{1}{2} + \sqrt{z}\right) + (-\sqrt{z})\left(\frac{1}{2} - \sqrt{z}\right)$$

Problem 3. Find (linear, homogenous) recurrence equations and initial conditions for

(a) $a_n = (n+1)2^n + (n-1)(-2)^n$

(b) $a_n = 1 + z^n$ where $z \in \mathbb{R} \setminus \{0\}$

Solution.

(a) Roots: 2 and -2, $(n+1)$ and $(n-1)$ are both linear $\to$ multiplicity of 2 for both roots. Thus, $h(x) = (x-2)^2(x+2)^2 = x^4 - 8x^2 + 16$. Therefore, the recurrence relation is:

$$a_n - 0a_{n-1} - 8a_{n-2} + 0a_{n-3} + 16a_{n-4} = 0$$

(b) Roots: 1 and z, both have multiplicity of 1. Thus, $h(x) = (x-1)(x-z) = x^2 - (1+z)x + z$. Therefore, the recurrence relation is:

$$a_n - (1+z)a_{n-1} + za_{n-2} = 0$$

9.7 Tutorial 7 (Midterm Week)

Problem 1. Determine the following coefficient. $[x^{2n}](x/(1-x))^n$.

Solution.

Problem 2. Let S be the class of binary strings in which every block has length of at most 6, every block of zeros has even length, and every block of ones has odd length. Find an unambiguous expression for S, and use it to compute $\Phi_S(x)$.

Solution.

Decomposition strategy (from course notes):

$$\{0\}^*(\{1\}\{1\}^*\{0\}\{0^*\})^*\{1\}^*$$

$B_0 = \{\varepsilon, 00, 0000, 000000\}$

$B_1 = \{\varepsilon, 1, 111, 11111\}$

$$S = B_0(B_1 \setminus \varepsilon \; B_0 \setminus \varepsilon)^* B_1$$
$$\Phi_{B_0}(x) = 1 + x^2 + x^4 + x^6$$
$$\Phi_{B_1}(x) = 1 + x + x^3 + x^5$$
$$\Phi_S(x) = \frac{\Phi_{B_0}(x)\Phi_{B_1}(x)}{1 - (x^2 + x^4 + x^6)(x + x^3 + x^5)}$$

Problem 3. For each n, let a_n denote the number of compositions of n where every part is even, and the number of parts is a multiple of 3. Find an explicit formula for a_n.

Solution.

Let $A = \{(\alpha_1, \alpha_2, \ldots, \alpha_{3m}) : \alpha_i \in \mathbb{N}_{\text{even}}\,, m \in \mathbb{N}\}$.

$$A = \bigcup_{m \geqslant 0} (\mathbb{N}_{\text{even}})^{3m}(x)$$

$$\Phi_{\mathbb{N}_{\text{even}}} = x^2 + x^4 + \cdots = \frac{x^2}{1 - x^2}$$

$$\begin{aligned}
\Phi_A(x) &= \sum_{m \geqslant 0} \Phi_{\mathbb{N}_{\text{even}^{3m}}}(x) && \text{by Sum Lemma} \\
&= \sum_{m \geqslant 0} \Phi_{\mathbb{N}_{\text{even}}}(x)^{3m} && \text{by Product Lemma} \\
&= \sum_{m \geqslant 0} \left(\frac{x^2}{1 - x^2}\right)^{3m} \\
&= \sum_{m \geqslant 0} \left(\frac{x^6}{(1 - x^2)^3}\right)^{m} \\
&= \frac{1}{1 - \frac{x^6}{(1-x^2)^3}} \\
&= \frac{(1 - x^2)^3}{(1 - x^2)^3 - x^6}
\end{aligned}$$

Thus,

$$a_n = [x^n]\frac{(1 - x^2)^3}{(1 - x^2)^3 - x^6}$$

Problem 4. Solve the linear recurrence relation defined by $a_n = 5a_{n-1} - 6a_{n-2}$, with initial conditions $a_0 = 2$ and $a_1 = 5$.

Solution.

Textbook Problem.

$$S_k = \{(A, B) : A, B \subseteq \{1, \ldots, n\}, |A| = |B| = m, |A \cap B| = k\}$$
$$T_k = \{(X, Y, Z) : X, Y, Z \subseteq \{1, \ldots, n\}, |X| = k, |Y| = |Z| = m - k, X \cap Y = X \cap Z = Y \cap Z = \varnothing\}$$

Define a bijection $f : S_k \to T_k$ and it's inverse.

$$f(A, B) = (A \cap B, A \setminus (A \cap B), B \setminus (A \cap B))$$

9.8 Tutorial 7

Problem 1. Determine the number of vertices and edges of each of the following graphs.

$G(n,k)$ for each n and k. For integers n and k, let $G(n,k)$ be the graph whose vertices are the k-element subsets of $\{1,\ldots,n\}$, where two vertices A and B are adjacent if $|A\cap B|\leqslant 2$.

(a) The graph G_1 whose vertices are the 4-element subsets of $\{1,\ldots,8\}$, where two vertices A and B are adjacent if and only if $|A\cap B|\leqslant 2$.

(b) The graph G_2 whose vertices are the binary strings of length n, where two vertices are adjacent if and only if they differ in exactly two positions.

Solution. The number of vertices are 2^n. Let $s\in V(G_2)$. How many other elements is s adjacent to?

$$\deg(s)=\binom{n}{2}$$

Handshake Lemma:

$$2|E|=\sum_{s\in V(G)}\deg(s)=2^n\binom{n}{2}$$

Thus,

$$|E|=2^{n-1}\binom{n}{2}$$

(c) The graph G_3 with vertex set $\{1,2,3,4,5\}\times\{1,2,3,4,5\}$, where two vertices (x,y) and (x',y') are adjacent if and only if $(x-x')^2+(y-y')^2\leqslant 2$.

Solution. The number of vertices are $|V(G_3)|=5\times 5=25$. Let $(x,y)\in V(G_3)$. What vertices are adjacent to (x,y)?

$$(x+1,y),(x-1,y),(x+1,y-1),(x-1,y-1),(x,y-1),(x+1,y+1),(x-1,y+1),(x,y+1)$$

If $2\leqslant x,y\leqslant 4$,

$$\deg((x,y))=8$$

If $x\in\{1,5\}$, $2\leqslant y\leqslant 5$,

$$\deg((x,y))=5$$

If $x,y\in\{1,5\}$,

$$\deg((x,y))=3$$

Using the Handshake Lemma,

$$\begin{aligned}2|E|&=\sum_{v\in V(G_3)}\deg(v)\\&=8(3\times 3)+5(2\times 3\times 2)+3(2\times 2)\\&=72+60+12\\&=144\end{aligned}$$

Thus,

$$|E|=72$$

Problem 2. Which of the graphs in the previous question are connected? Give a proof either way.

(b) **Solution.** $n=4$, $\binom{n}{2}=\binom{4}{2}=6$. We know 1010 is adjacent to $1001,0011,0000,1100,0110,1111$. Note that the parity is the same in all.

Connected: $\forall x,y\in V(G)$, there exists a path between x and y in G.

Consider the path between 0001 and 1101.

Changing two bits will always leave the parity the same (since we either add 2 to the sum, subtract 2, or add 1 and subtract 1). Therefore, there is no path between a vertex of even parity and a vertex with odd parity. Thus, G_2 is not connected.

(c) **Solution.** Claim: G_3 is connected. Suppose for a contradiction that G_3 is not connected. Then, there exists a $(x, y), (x', y') \in V(G_3)$ such that there is not path between them. WLOG, $x \leqslant x'$ and $y \leqslant y'$ (invert one of the following sequences in each case we have $>$)

$$x, x+1, \dots, x' \qquad ; \qquad y, y+1, \dots, y'$$

$$(x, y) \sim (x+1, y) \sim \cdots \sim (x', y) \sim (x', y+1) \sim (x', y')$$

Contradiction.

Problem 3. Prove that every graph on at least two vertices has two vertices of the same degree.

Solution. Suppose for a contradiction that $V = \{v_1, \dots, v_k\}$ have different degrees. Say d_i is the degree of v_i for each $i \in [1, k]$. WLOG,

$$d_1 < d_2 < \cdots < d_k$$

You can assume that $d_1 \geqslant 1$.

$$d_1 \geqslant 1 \implies d_i \geqslant i \qquad \forall i$$

$$d_k \geqslant k \implies v_k \text{ is adjacent to } k \text{ vertices, but there are only } k-1 \text{ available}$$

contradiction. If $d_1 = 0$, then it doesn't affect the degrees of other vertices so we can remove it from G, and just look at

$$v_2, \dots, v_k$$